HOMEWORK 12 – ANSWERS TO (MOST) PROBLEMS

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Section 6.1: Areas between curves

6.1.1. $\int_0^4 (5x - x^2) - x dx = \int_0^4 4x - x^2 dx = \boxed{\frac{32}{3}}$

6.1.3. $\int_{-1}^1 e^y - (y^2 - 2) dy = \boxed{e - e^{-1} + \frac{10}{3}}$

6.1.13. $\int_{-3}^3 (12 - x^2) - (x^2 - 6) dx = \int_{-3}^3 18 - 2x^2 dx = = \boxed{72}$
(points of intersection are $x = \pm 3$)

6.1.21. To find the points of intersection, solve:

$$\begin{aligned}
\tan(x) =& 2\sin(x) \\
\frac{\sin(x)}{\cos(x)} =& 2\sin(x) \\
\sin(x) =& 2\sin(x)\cos(x) \\
\sin(x)(1 - 2\cos(x)) =& 0
\end{aligned}$$

which implies either $\sin(x) = 0$, that is $x = 0$, or $\cos(x) = \frac{1}{2}$, that is, $x = \pm\frac{\pi}{3}$.

Hence, if you draw a good picture, you'll see that we need to find:

$$\int_{-\frac{\pi}{3}}^0 \tan(x) - 2\sin(x) dx + \int_0^{\frac{\pi}{3}} 2\sin(x) - \tan(x) dx$$

But by symmetry (see your picture), both of those integrals are equal to each other, and therefore:

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\begin{aligned}
A =& 2\left(\int_0^{\frac{\pi}{3}} 2\sin(x) - \tan(x)dx\right) \\
=& 2\left[-2\cos(x) - \ln(\sec(x))\right]_0^{\frac{\pi}{3}} \\
=& 2\left(-2\frac{1}{2} - \ln(2) + 2 - \ln(1)\right) \\
=& 2\left(-1 - \ln(2) + 2 - 0\right) \\
=& 2\left(1 - \ln(2)\right) \\
=& 2 - 2\ln(2)
\end{aligned}
$$

6.1.42. $\int_{-\frac{1}{2}}^{\frac{1}{2}} 1 - |y| - 2y^2 dy = \int_{-\frac{1}{2}}^{0} 1 + y - 2y^2 dy + \int_0^{\frac{1}{2}} 1 - y - 2y^2 dy = -\frac{7}{24} + \frac{7}{24} = \boxed{\frac{7}{6}}$.
(to find the points of intersection, solve $2y^2 = 1 - |y|$, and split up into the two cases $y \geq 0$ and $y < 0$). Also, it might help to notice that your function is even, so you really only care about the case where $y \geq 0$.

6.1.43. Here $n = 5$, and $D \approxeq 2(f(1) + f(3) + f(5) + f(7) + f(9)) = \frac{2}{60}(2 + 6 + 9 + 11 + 12) = \boxed{117\frac{1}{3}}$, where $f(x) = v_K - v_C$ (notice that $v_K \geq v_C$ throughout the race!)

6.1.51. The first region has area equal to $\int_0^b 2\sqrt{y}dy = \frac{4}{3}b^{\frac{3}{2}}$ (notice that we're integrating with respect to y, and $y = x^2 \Leftrightarrow y = \pm\sqrt{x}$. Also, draw a picture to see why we have an extra factor of 2 in the integral). The second region has area equal to $\int_b^4 2\sqrt{y}dy = -\frac{4}{3}b^{\frac{3}{2}} + \frac{32}{3}$, so to solve for b, we need to set those two areas equal:

$$\frac{4}{3}b^{\frac{3}{2}} = -\frac{4}{3}b^{\frac{3}{2}} + \frac{32}{3} \Leftrightarrow \frac{8}{3}b^{\frac{3}{2}} = \frac{32}{3} \Leftrightarrow b^{\frac{3}{2}} = 4 \Leftrightarrow b = 4^{\frac{2}{3}}$$

Section 6.2: Volumes

Note: In case you're confused by what I mean with K, Outer, Inner, etc., make sure to check out the 'Volumes'-Handout on my website (which is the handout given in section)

6.2.6. Disk method, $K = 0$, $x = e^y$, so $\int_1^2 \pi(e^y)^2 dy = \int_1^2 \pi(e^{2y})dy = \boxed{\frac{\pi}{2}(e^4 - e^2)}$

6.2.13. Washer method, $K = 1$, Outer $= (3) - 1 = 2$, Inner $= (1 + \sec(x)) - 1 = \sec(x)$, Points of intersection $\pm\frac{\pi}{3}$, so:

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \pi(2^2 - \sec^2(x))dx = \pi(4\frac{2\pi}{3} - \tan(\frac{\pi}{3}) + \tan(\frac{-\pi}{3})) = \pi(\frac{8\pi}{3} - 2\sqrt{3}) = 2\pi\left(\frac{4}{3}\pi - \sqrt{3}\right)$$

6.2.17. Washer method, $K = -1$, and notice $y = x^2 \Leftrightarrow x = \sqrt{y}$ (in this case $x \geq 0$), Outer $= \sqrt{y} - (-1) = \sqrt{y} + 1$, Inner $= y^2 - (-1) = y^2 + 1$, Point of intersection $y = 0$ and $y = 1$, so:

$$\int_0^1 \pi(\sqrt{y} + 1)^2 - (y^2 + 1)^2 dy = \frac{29\pi}{30}$$

6.2.49. Disk method, $K = 0$, $\int_{r-h}^{r} \pi(\sqrt{r^2 - y^2})^2 dy = \int_{r-h}^{r} \pi(r^2 - y^2)dy \boxed{\pi h^2\left(r - \frac{1}{3}h\right)}$ (use the fact that $x^2 + y^2 = r^2$, and solve for y)

6.2.55. $A(x) = \frac{1}{2}L^2 = \frac{1}{2}(\frac{b}{\sqrt{2}})^2 = \frac{1}{4}b^2 = \frac{1}{4}(2y)^2 = y^2 = \frac{36-9x^2}{4} = 9 - \frac{9}{4}x^2$ (here L is the length of a side of the triangle, and $b = 2y$ is the hypotenuse) so $V = \int_{-2}^{2}\left(9 - \frac{9}{4}x^2\right)dx = \boxed{24}$ (you get the endpoints by setting $y = 0$ in $9x^2 + 4y^2 = 36$)

6.2.65. The point is to draw a very good picture! Make one sphere have center $(0, -\frac{r}{2})$ in the $xy-$plane and the other one have center $(0, \frac{r}{2})$. Then the volume is really the volume of two pieces of equal volume, let's focus on $x \geq 0$ only! Then, using the disk method, you get:

$$V = 2\int_0^{\frac{r}{2}} \pi\left(\sqrt{r^2 - \left(x + \frac{r}{2}\right)^2}\right)^2 dx = 2\pi\int_0^{\frac{r}{2}} r^2 - \left(x + \frac{r}{2}\right)dx = \frac{5\pi r^3}{12}$$

(here we used the fact that $(x + \frac{r}{2})^2 + y^2 = r^2$, and solved for y. This looks a bit strange, but remember that your height is really on the left sphere, not on the right one!)

6.2.68. This is **much** easier with the shell method of section 6.3. Here $K = 0$, $f(x) = \sqrt{R^2 - x^2}$ (since $x^2 + y^2 = R^2$), and so $\int_r^R 2\pi x\sqrt{R^2 - x^2}dx = \boxed{\frac{2\pi}{3}\left(R^2 - r^2\right)^{\frac{3}{2}}}$ (use the substitution $u = R^2 - x^2$)

Section 6.3: Volumes by cylindrical shells

6.3.2. $\int_0^{\sqrt{\pi}} 2\pi x \sin(x^2)dx = \boxed{2\pi}$ (use the substitution $u = x^2$)

6.3.13. Shell method: $K = 0$, $|y - 0| = y$, Outer $= 2$, Inner $= 1+(y-2)^2$, Points of intersection $y = 1$, $y = 3$, so $\int_1^3 2\pi y(2-(1+(y-2)^2))dy = \int_1^3 2\pi y(1-(y-2)^2))dy = \boxed{\frac{16\pi}{3}}$.

6.3.15. Shell method: $K = 2$, $|x - 2| = 2 - x$, Outer $= x^4$, Inner $= 0$, $\int_0^1 2\pi(2 - x)(x^4)dx = \boxed{\frac{7\pi}{15}}$

6.3.19. Shell method: $K = 1$, $|y - 1| = 1 - y$, Outer $= 1$, Inner $= \sqrt[3]{y}$, $\int_0^1 2\pi(1 - y)(1 - \sqrt[3]{y})dy = \boxed{\frac{5\pi}{14}}$

6.3.46. Shell method: $K = 0$, $|x| = x$, Outer $= \sqrt{r^2 - (x - R)^2}$ (use the fact that $(x-R)^2+y^2 = r^2$), Innter $= -\sqrt{r^2 - (x - R)^2}$, so $\int_{R-r}^{R+r} 2\pi x 2\sqrt{r^2 - (x - R)^2}dx = \boxed{\pi^2 R r^2}$ (use the substitution $u = x - R$, and remember what you did in 5.5.73)

6.3.48. Shell method: $K = 0$, $|x| = x$, Outer $= 2\sqrt{R^2 - x^2}$ (use the fact that $x^2 + y^2 = R^2$), Inner $= 0$,

$$\int_r^R 2\pi x(2\sqrt{R^2 - x^2})dx = \frac{4\pi}{3}(R^2 - r^2)^{\frac{3}{2}} = \frac{4\pi}{3}\left(\left(\frac{h}{2}\right)^2\right)^{\frac{3}{2}} = \frac{4\pi}{3}\frac{h^3}{8} = \frac{\pi h^3}{6}$$

(use the substitution $u = R^2 - x^2$, and the fact that $r^2 + (\frac{h}{2})^2 = R^2$ by the Pythagorean theorem)